

# On Supermultiplet Twisting and Spin-Statistics

T. Hübsch

Department of Physics & Astronomy, Howard University, Washington, DC  
 Department of Physics, University of Central Florida, Orlando FL  
 thubsch@howard.edu

## ABSTRACT

Twisting of off-shell supermultiplets in models with 1+1-dimensional spacetime has been discovered in 1984, and was shown to be a generic feature of off-shell representations in worldline supersymmetry two decades later. It is shown herein that in all supersymmetric models with spacetime of four or more dimensions, this type of twisting, if nontrivial, necessarily maps regular (non-ghost) supermultiplets to ghost supermultiplets.

PACS: 11.30.Pb, 12.60.Jv

*Most every ghost will subsist in the twist  
 of a thought once distraught, but dismissed.  
 — Algernon Eduard Beaugh*

## 1 Introduction, Results and Summary

The *twisted* variant of the well-known chiral off-shell supermultiplets and superfield in 1+1-dimensional (2,2)-supersymmetric field theories was discovered in 1984 [1]. Ref. [2] then proved that the combination of chiral and twisted chiral superfields provides the singularly exceptional means of constructing worldsheet models with non-Kähler target spaces, making them *usefully inequivalent* [3]. Refs. [4,5,6] show that this *twisting* amounts to changing the sign of an odd number of real (Hermitian) supercharge components. The results of Refs. [7,8,9,10] imply that the twisted variant of a supermultiplet is inequivalent to the original whenever the number  $N$  of supercharge components<sup>1</sup> is divisible by four and the twisting changes the sign of an odd number of real supercharges involved in an odd number of  $\mathbb{Z}_2$ -projections used to obtain the supermultiplet from an *intact*<sup>2</sup> supermultiplet.

In all spacetimes of dimension four or more, all spinors—and so also the supercharges—automatically have the number of components divisible by four. Nevertheless, the phenomenon of supermultiplet *twisting* has never been observed in higher-dimensional supersymmetric models: It is thus natural to ask: “Where have all the twisted supermultiplets gone?” Herein, we prove:

**Theorem 1.1** (*Spin-Statistics of Twisting*) *Every adinkraic<sup>3</sup> off-shell supermultiplet of  $N$ -component supersymmetry with no central charge in  $d$ -dimensional spacetime has a twisted*

<sup>1</sup> Herein,  $N$  counts all individual spinor components; in spacetime dimensions  $d \geq 4$ ,  $N \equiv 0 \pmod{4}$ .

<sup>2</sup> Herein, the adjective “**intact**” will stand for “unprojected, unconstrained, ungauged,” so intact supermultiplets are the same as defined by the standard Salam-Strathdee superfields [11,12,13].

<sup>3</sup> A worldline supermultiplet is “**adinkraic**” if it admits a basis of component fields such that every supercharge component turns every component field into precisely one other component field or its derivative [6]. By extension, a higher-dimensional supermultiplet is adinkraic if its worldline dimensional reduction is.

variant. If not equivalent to the original by field redefinitions, this twisted variant must consist purely of ghost<sup>4</sup> component fields for  $d \geq 4$ .

For  $d \leq 2$ ,  $Spin(1, d-1)$  is abelian and the component fields may be chosen at will to be either regular or ghost fields. The intermediate,  $d = 3$  case is special owing to the possible inclusion of anyons and will not be discussed herein. This result then prompts the following:

**Conjecture 1.1** *Extending by linearity, the statement of theorem 1.1 applies to all supermultiplets that may be constructed from adinkraic supermultiplets either by using superderivative constraints and equivalence classes as in the semi-indefinite sequences of Ref. [6,14], or by an adaptation of Weyl's iterative construction of representations, such as in Ref. [15].*

The remainder of this introduction provides the necessary notation and conventions for worldline  $N$ -component supersymmetry with no central charge, and section 2 discusses the worldline reduction of supermultiplet twisting [1,2,6,8,9,10]. Section 3 then presents the proof of Theorem 1.1 and discusses some extensions and generalizations that motivate the Conjectures 1.1 and 3.1. Section 4 closes with a few concluding comments

**Conventions:** The worldline  $N$ -extended supersymmetry algebra with no central charge is

$$\left\{ \begin{array}{l} \{Q_I, Q_J\} = 2\delta_{IJ} H, \quad [H, Q_I] = 0, \\ Q_I^\dagger = Q_I, \quad H^\dagger = H, \end{array} \right\} \quad I, J = 1, 2, \dots, N, \quad (1)$$

where  $H = i\hbar\partial_\tau$ , but we adopt  $\hbar$  as one of the units and no longer write it. Superspace methods<sup>5</sup> also involve the use of superderivatives:

$$\left\{ \begin{array}{l} \{D_I, D_J\} = 2i\delta_{IJ} \partial_\tau, \quad [\partial_\tau, D_I] = 0, \\ D_I^\dagger = -D_I, \quad \{Q_I, D_J\} = 0, \end{array} \right\} \quad I, J = 1, 2, \dots, N, \quad (2)$$

such that

$$Q_I = iD_I + 2i\delta_{IJ}\theta^J \partial_\tau, \quad \text{and} \quad D_I = -iQ_I + 2i\delta_{IJ}\theta^J H, \quad (3)$$

where  $\theta^I$  provide the fermionic extension of (space)time into superspace. When applied on superfields (general functions over superspace), the  $D_I$  act as left-derivatives while the  $Q_I$  act as right-derivatives. Therefore, and because of the relationship (3), we have the identities:

$$Q_I(b) := iD_I \mathbb{B} \quad \text{and} \quad Q_I(f) := -iD_I \mathbb{F}, \quad (4)$$

where  $b := \mathbb{B}$  is an arbitrary bosonic functional-differential expression,  $f := \mathbb{F}$  an arbitrary fermionic functional-differential expression, and  $\mathbb{B}$  and  $\mathbb{F}$  are the appropriate superfield functional-superderivative expressions that define  $b$  and  $f$ , respectively.

<sup>4</sup> Herein, any field with the commuting/boson  $\leftrightarrow$  Lorentz-tensor and anticommuting/fermion  $\leftrightarrow$  Lorentz-spinor spin-statistics correspondence will be called “**regular**,” whereas fields with the opposite spin-statistics correspondence will be called “ghosts,” even if their role in any particular model may turn out to be quite different.

<sup>5</sup> Although it is by no means necessary to use superspace methods, we find it convenient to do so; this incurs no loss of generality since supersymmetry implies the existence of superspace [16].

## 2 Supermultiplet Twisting

The original example of supermultiplet twisting in Refs. [1,2] pertains to the chiral and twisted chiral superfield pair. By rewriting the superderivatives from the usual, Weyl complex notation into a suitable real basis, these are defined as:

$$\text{chiral:} \quad [D_1 - iD_3]\Phi = 0 = [D_2 - iD_4]\Phi, \quad (5)$$

$$\text{twisted chiral:} \quad [D_1 - iD_3]\Xi = 0 = [D_2 + iD_4]\Xi. \quad (6)$$

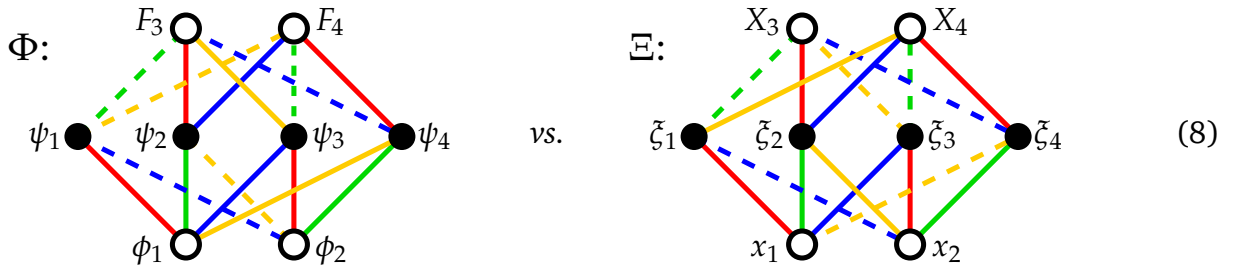
The *supermultiplet twisting*  $\Phi \rightarrow \Xi$  is thus equivalent to the sign-change

$$D_4 \rightarrow -D_4, \quad \text{whereby also} \quad Q_4 \rightarrow -Q_4, \quad (7)$$

which cannot be compensated by (component) field redefinitions. Evidently, such a sign-change is possible only if the individual supercharge components  $Q_I$  may be redefined in a Lorentz-covariant way, which is true only for  $d \leq 2$ , i.e., on the worldsheet and on the worldline. In spacetimes of dimension  $d > 2$ , the Lorentz group  $Spin(1, d-1)$  is non-abelian, the minimal spinor representations have an even number of real components<sup>6</sup>, and changing the sign of an odd number of supercharge components violates Lorentz symmetry.

Even in spacetimes of  $d \leq 2$  dimensions where the redefinition  $D_4 \rightarrow -D_4$  provides the isomorphism  $\Phi \xrightarrow{\sim} \Xi$ , this isomorphism cannot be employed in models that include interactions between  $\Phi$  and  $\Xi$ : these supermultiplets swap their structures and so retain their relative difference.





With a judicious definition of real component fields and adopting the conventions of Refs. [6, 7, 8, 9, 10], these supermultiplets may be depicted by the Adinkras:



Adinkra edges represent supersymmetry transformations with edge-colors labeling the distinct supercharges: red =  $Q_1$ , green =  $Q_2$ , blue =  $Q_3$ , yellow =  $Q_4$ ; bosons (fermions) are labeled by white (black) nodes; Table 1 provides a precise dictionary applicable in the dimensional reduction of the supermultiplets to their worldline. The  $D_4 \rightarrow -D_4$  ( $Q_4 \rightarrow -Q_4$ ) twisting is then evident on comparing the two Adinkras (8): they differ solely in the dashedness of the yellow edges representing the  $Q_4$ -transformations.

The same Adinkra may in principle also be used to depict the supersymmetry transformations in higher-dimensional spacetimes—provided the Adinkra is consistent with the additional generators of the higher-dimensional Poincaré group,  $Spin(1, d-1) \ltimes Tr(1, d-1)$ . Conditions for this consistency and therefore dimensional extension to supersymmetry in higher-dimensional spacetimes are being actively investigated [17, 18, 19, 15, 20].

<sup>6</sup> In fact, only in  $d = 3$  have minimal spinors only two real components; in dimensions  $d \geq 4$ , minimal spinors have number of real components divisible by four.

Adinkra	Q-action	Adinkra	Q-action
	$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} i\dot{\phi}_A \\ \psi_B \end{bmatrix}$		$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} -i\dot{\phi}_A \\ -\psi_B \end{bmatrix}$
	$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} \dot{\psi}_B \\ i\phi_A \end{bmatrix}$		$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} -\dot{\psi}_B \\ -i\phi_A \end{bmatrix}$

Edges may be drawn in the  $I^{\text{th}}$  color instead of being labeled by  $I$ .

**Table 1:** Adinkras assign: (white/black) vertices to (boson/fermion) component fields;  $I^{\text{th}}$  edge color to  $Q_I$ ; solid/dashed edge to  $\pm 1$  signs in the tabulated supersymmetry action; nodes are drawn at heights equal to the engineering dimension of the depicted component (super)field.

**Cycle Parity:** Besides depicting the system of  $(4+4) \times 4 = 32$  explicit component-wise supersymmetry transformation equations in an intuitive but 1–1 faithful way, the Adinkras (8) easily distinguish  $\Phi$  from  $\Xi$ ; to show this, some observations made in Refs. [8,9,21] are formalized:

**Definition 2.1** Let  $\mathbf{c}$  denote an ordered collection of distinct edge-colors. Within an Adinkra, the **cycle parity** of any cycle (closed path)  $\mathcal{C}_{\mathbf{c}}$  of  $\mathbf{c}$ -colored edges (one edge for every color in  $\mathbf{c}$ ) is the product

$$CP(\mathcal{C}_{\mathbf{c}}) := (-1)^{\varepsilon_*} \cdot (-1)^{\varepsilon_{\mathbf{c}}(\mathcal{C}_{\mathbf{c}})} \cdot \prod_{\text{edge} \in \mathcal{C}_{\mathbf{c}}} (-1)^{\varepsilon(\text{edge})} \quad (9)$$

where  $\varepsilon_* = 0$  (1) if  $\mathcal{C}_{\mathbf{c}}$  starts from a bosonic (fermionic) node<sup>7</sup>,  $\varepsilon_{\mathbf{c}}(\mathcal{C}_{\mathbf{c}}) = 0$  (1) if  $\mathcal{C}_{\mathbf{c}}$  follows an even (odd) permutation of  $\mathbf{c}$ , and  $\varepsilon(\text{edge}) = 0$  (1) if the edge is solid (dashed).

The Adinkras (8) make it easy to compute ( $\mathbf{c} = \{\text{red, green, blue, yellow}\}$ )

$$CP(\mathcal{C}_{\mathbf{c}}) = +1, \quad \forall \mathcal{C}_{\mathbf{c}} \subset \mathbf{A}(\Phi), \quad \text{and} \quad CP(\mathcal{C}_{\mathbf{c}}) = -1, \quad \forall \mathcal{C}_{\mathbf{c}} \subset \mathbf{A}(\Xi). \quad (10)$$

For any *adinkraic* supermultiplet,  $CP_{\mathbf{c}}(\mathbf{A})$  equals the quantity  $\chi_0$  defined in Ref. [22], is closely related to the “Diamonds and Bow-ties” theorem in Ref. [20];  $CP_{\mathbf{c}}(\mathbf{A})$  is also well-defined for the Adinkra-like graphs shown in Refs. [28,14,23] as depicting non-adinkraic supermultiplets.

**Cycle-Independence of  $CP(\mathcal{C}_{\mathbf{c}})$ :** We now prove that  $CP(\mathcal{C}_{\mathbf{c}})$  is actually independent of the particular cycle  $\mathcal{C}_{\mathbf{c}}$  used to compute it, and is a characteristic of the whole Adinkra,  $\mathbf{A}$ .

**Proof:** The values (10) may in fact be computed directly from the superspace definitions (5)–(6), as follows: Denoting the real and imaginary parts of the chiral superfield as  $\Phi = \mathbb{A} + i\mathbb{B}$ , the complex system of superderivative constraints (5) is seen to be equivalent to the real system

$$(D_1\mathbb{A} + D_3\mathbb{B}) = 0 = i(D_1\mathbb{B} - D_3\mathbb{A}), \quad (11a)$$

$$(D_2\mathbb{A} + D_4\mathbb{B}) = 0 = i(D_2\mathbb{B} - D_4\mathbb{A}), \quad (11b)$$

<sup>7</sup>The  $(-1)^{\varepsilon_*}$  factor in the definition of  $CP_{\mathbf{c}}$  is the same relative sign between the two expressions in Eqs. (4).

where elimination of  $\mathbb{B}$ , and then in turn of  $\mathbb{A}$  is seen to imply, respectively,

$$\mathbb{D}_{IJ}^+ \mathbb{A} = 0, \quad \text{and} \quad \mathbb{D}_{IJ}^+ \mathbb{B} = 0, \quad (12a)$$

with

$$\mathbb{D}_{IJ}^\pm := [D_I D_J \pm \frac{1}{2} \varepsilon_{IJ}^{KL} D_K D_L], \quad (12b)$$

so that the projection constraint (12a) equally applies to  $\Phi = (\mathbb{A} + i\mathbb{B})$ . A second application of  $\mathbb{D}_{IJ}^+$  then produces:

$$[\partial_\tau^2 + D_1 D_2 D_3 D_4] \Phi = 0, \quad (12c)$$

$$\stackrel{(4)}{\implies} \quad \textcolor{red}{Q}_1 \textcolor{green}{Q}_2 \circ \textcolor{blue}{Q}_3 \circ \textcolor{orange}{Q}_4 \circ \Phi = +(H^2 \Phi). \quad (12d)$$

Compare this with the analogous computation for  $\Xi$ , which yields

$$\mathbb{D}_{IJ}^- \Xi = 0, \quad (13a)$$

$$\Rightarrow [\partial_\tau^2 - D_1 D_2 D_3 D_4] \Xi = 0, \quad (13b)$$

$$\stackrel{(4)}{\implies} \quad \textcolor{red}{Q}_1 \circ \textcolor{green}{Q}_2 \circ \textcolor{blue}{Q}_3 \circ \textcolor{orange}{Q}_4 \Xi = -(H^2 \Xi). \quad (13c)$$

Thus:

1. The 4-color path depicting the  $\textcolor{red}{Q}_1 \circ \textcolor{green}{Q}_2 \circ \textcolor{blue}{Q}_3 \circ \textcolor{orange}{Q}_4$  action in the Adinkra is closed precisely if the supermultiplet satisfies a superderivative system of constraints akin to (12a) and (13a).
2. The relative signs in the superderivative binomial operators (12a) and (12c) equal the sign on the right-hand side of (12d) and the sign computed for  $\Phi$  by definition 2.1; the corresponding signs in (13a), (13b) and (13c) equal the sign for  $\Xi$  by the same definition.

The lowest components of both  $\Phi$  and  $\Xi$  are complex bosons. The superspace  $\rightarrow$  (space)time projection of (12d) and (13c) immediately reads off the value of  $CP(\mathcal{C}_{\mathbf{c}}) = +1$ , i.e.,  $CP(\mathcal{C}_{\mathbf{c}}) = -1$ , where  $\mathcal{C}_{\mathbf{c}}$  is the 4-cycle of edges starting and ending at either the real or the imaginary part of the lowest bosonic component field, and colored in a permutation of  $\mathbf{c} = \{\text{red, green, blue and yellow}\}$ . Since distinct  $D_I$ 's and  $Q_I$ 's all anticommute, the  $D_I$ - and the  $Q_I$ -monomials in (12c), (12d), (13b) and (13c) may be freely permuted, keeping track of the resulting change in the relative sign by  $(-1)^{\varepsilon_{\mathbf{c}}(\mathcal{C})}$ . Finally, all other component fields are computed by projecting a  $D_{[I_1} \cdots D_{I_n]}$ -superderivative of  $\Phi$ , i.e.,  $\Xi$ . Since

$$D_{[I_1} \cdots D_{I_n]} [H^2 \pm \textcolor{red}{D}_1 \textcolor{green}{D}_2 \textcolor{blue}{D}_3 \textcolor{orange}{D}_4] = [H^2 \pm (-1)^n \textcolor{red}{D}_1 \textcolor{green}{D}_2 \textcolor{blue}{D}_3 \textcolor{orange}{D}_4] D_{[I_1} \cdots D_{I_n]}, \quad (14)$$

the sign computed starting from a fermionic component field (odd  $n$ ) is opposite of that computed by starting from a bosonic component field (even  $n$ ). Generalizations of arbitrary cycles  $\mathcal{C}_{\mathbf{c}}$  will simply replace the quartic  $D_I$ - and  $Q_I$ -monomials from the computations (12a)–(14) with the  $\mathcal{C}_{\mathbf{c}}$ -depicted  $D_I$ - and  $Q_I$ -monomials and their permutations.

Since the foregoing computation holds for the entire superfield/supermultiplet,  $CP_{\mathbf{c}}(\mathcal{C}_{\mathbf{c}})$  is independent of the particular cycle  $\mathcal{C}_{\mathbf{c}} \subset \mathbf{A}$ . Finally, if a superfield/supermultiplet satisfies no superderivative constraint system akin to (12a) or (13a), there is no cycle  $\mathcal{C}_{\mathbf{c}}$  of distinctly  $\mathbf{c}$ -colored edges in the Adinkra  $\mathbf{A}$  depicting the supermultiplet and the corresponding ‘default’ value must be set to  $CP_{\mathbf{c}}(\mathbf{A}) = 0$ . ✓

**Definition 2.2**  $CP_{\mathbf{c}}(\mathbf{A}) = CP(\mathcal{C}_{\mathbf{c}} \subset \mathbf{A})$  is a characteristic of the supermultiplet depicted by the Adinkra  $\mathbf{A}$ . If the Adinkra  $\mathbf{A}$  contains no  $\mathbf{c}$ -colored cycle,  $CP_{\mathbf{c}}(\mathbf{A}) := 0$ .

**Implications:** Systems of superderivative constraints akin to (12a) were used in Ref. [21] to construct a super-constrained superfield for each of  $\sim 10^{12}$  chromotopologies of Ref. [8], whereupon Theorem 7.6 of Ref. [6] constructs a superfield representation for every supermultiplet with that chromotopology. The particular superderivative monomial occurring in the induced superderivative constraint (12c) permits reading off the chromotopology of the so-defined superfield and supermultiplet. For the present purposes a comparison of Eqs. (12c) and (13b) makes it obvious that  $\Phi$  and  $\Xi$  satisfy complementary projections.

The classification work of Refs. [7,8,9,10] and the explicit construction in Ref. [21] imply that for each  $N$  there is a single intact ( $N$ -cubical) chromotopology, but a combinatorially growing number of (multiple)  $\mathbb{Z}_2$ -projections, where each projection is encoded by superderivative constraints akin to Eqs. (12a)–(12c) and Eqs. (13a)–(13b). Supermultiplets the chromotopology of which is a multiple  $\mathbb{Z}_2$ -projection will satisfy *multiple* independent projections akin to that in (12c) and (13b). Flipping the relative sign in *two* independent such projections is equivalent to changing the sign of *two* of the supercharges, which can always be compensated by judicious sign-changes in some of the component fields.

It then follows that every Adinkra with a chromotopology of a (multiple)  $\mathbb{Z}_2$ -projection of an  $N$ -cube has at most one pair of mutually twisted variants. In turn, the twisting in adinkraic supermultiplets cannot be compensated by a judicious component field redefinition precisely when  $N \equiv 0 \pmod{4}$  [24]. The original and the twisted supermultiplets are inequivalent precisely if the twisting changes the sign of an odd number of supercharges involved in an odd number of distinct projections. The vast majority of the  $\sim 10^{12}$  chromotopologies are projected and so do have inequivalent twisted variants.

Thus, all adinkraic worldline dimensional reductions of supermultiplets in spacetimes of dimension four and higher must have a twisted variant, most of which inequivalent to the original.

**Alternate Twisting:** The factor  $(-1)^{\varepsilon_*}$  in the definition 2.1 implies that the twisted variant of any given Adinkra  $\mathbf{A}$  is also obtained by swapping the spin-statistics (boson  $\leftrightarrow$  fermion) of all nodes, referred to as the ‘Klein-flip,’  $\mathbf{A}^K$ . That is to say,

$$CP_{\mathbf{c}}(\mathbf{A}^K) = -CP_{\mathbf{c}}(\mathbf{A}), \quad (15)$$

and supermultiplet twisting is *equivalent* to Klein-flipping the supermultiplet, up to some component field redefinitions.

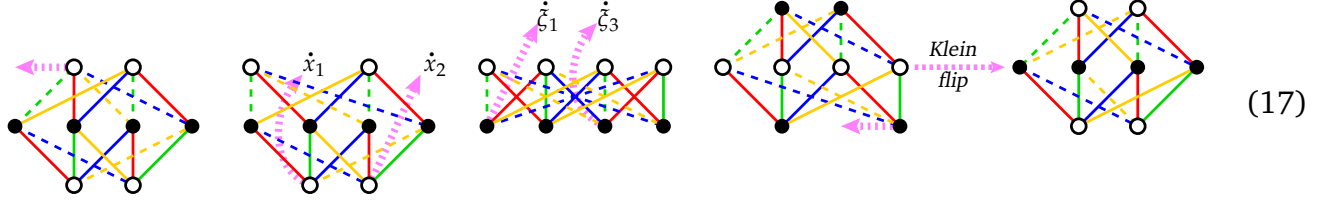
That is, this equivalence requires relating derivatives of fields, as is the case with:

$$\Xi \xrightarrow{K} \Phi : \quad \begin{cases} \phi_1 \leftrightarrow \tilde{\zeta}_2^K, & \phi_2 \leftrightarrow \tilde{\zeta}_4^K, & F_3 \leftrightarrow \dot{\tilde{\zeta}}_1^K, & F_4 \leftrightarrow \dot{\tilde{\zeta}}_3^K, \\ \psi_1 \leftrightarrow X_3^K, & \psi_2 \leftrightarrow \dot{x}_1^K, & \psi_3 \leftrightarrow X_4^K, & \psi_4 \leftrightarrow \dot{x}_2^K, \end{cases} \quad (16)$$

where, strictly speaking, the constant modes in  $\tilde{\zeta}_1^K(\tau)$ ,  $\tilde{\zeta}_3^K(\tau)$ ,  $x_1^K(\tau)$  and  $x_2^K(\tau)$  are dropped in the  $\Xi \xrightarrow{K} \Phi$  mapping. Such a mapping is therefore not a *strict homomorphism of off-shell supermultiplets* as defined in Ref. [9, Definition B.1], and  $\Phi$  and  $\Xi$  are not isomorphic as representations



of supersymmetry, but are each other's twisted variant. The replacements  $x_1 \rightarrow \dot{x}_1$  and  $x_2 \rightarrow \dot{y}_2$ , followed by  $\xi_1 \rightarrow \dot{\xi}_1$  and  $\xi_3 \rightarrow \dot{\xi}_3$  are easily depicted:



This illustrates how the assignments (16) produce, upon the final Klein flip (boson  $\leftrightarrow$  fermion, i.e., white  $\leftrightarrow$  black node assignment swap), the Adinkra of the chiral supermultiplet,  $\Phi$ , in (8).

Reading from the indicated operations on the second Adinkra (17), the local transformation  $x_1 \rightarrow (X_1 := \dot{x}_1)$  was dubbed “node-raising” [6]; its formal inverse,  $X_1 \rightarrow (x_1 := \int d\tau X_1)$  then corresponds to node-lowering. Both operations were introduced as “automorphic dualities” in Ref. [25]. Neither node-raising nor node-lowering are local isomorphisms: while node-lowering itself depicts the non-local field redefinition  $F \mapsto f := \int d\tau F$ , for node-raising it is the inverse of the depicted transformation  $F \mapsto \mathcal{F} := \dot{F}$  that is non-local. In addition, these operations are manifestly not Lorentz-covariant in  $(d \geq 3)$ -dimensional spacetime, and even on the worldsheet  $(d = 2)$  some instances of such field redefinitions are obstructed [19], essentially since the  $d \geq 2$  generalizations  $\partial_\tau \rightarrow \partial_\mu$  and  $\int d\tau \rightarrow \int dx^\mu$  are not Lorentz-scalars.

The operation (16)–(17) thus effectively performs the *supermultiplet twisting* without re-defining the supercharges. Its analogue was employed in Ref. [26] to construct the twisted variant of the so-called *ultra-multiplet* [27] of 8-component worldline supersymmetry. It is then

**Definition 2.3** (*Supermultiplet Twisting*) *Given any off-shell supermultiplet, its twisted variant is defined to be the supermultiplet obtained by the boson  $\leftrightarrow$  fermion Klein-flip.*

First and foremost, when accompanied by appropriate (node-raising/lowering) field redefinitions as done in (17), this definition of supermultiplet twisting fully agrees with all known cases of supermultiplet twisting in the literature; see Refs. [1,2,4,5,6,21], for example. Unlike the individual supercharge component sign change (7), this (re)definition of supermultiplet twisting is perfectly applicable to all supersymmetry in spacetimes of all dimensions. Finally, as the absence of restrictions on extendedness of supersymmetry or the dimensionality and signature of spacetime implies, definition 2.3 indeed applies without any such restrictions. This follows since the Klein-flip does not change the Poincaré representations that the supercharge components, the bosonic and the fermionic component fields respectively span: given a supermultiplet of  $N$ -extended supersymmetry in any spacetime and for any admissible  $N$ , its twisted variant specified in definition 2.3 is also an off-shell supermultiplet of the same supersymmetry.

We thus adopt the (re-)definition 2.3 for supermultiplet twisting *instead* of the supercharge component sign-change (7), which is limited to worldline and worldsheet supersymmetry.

### 3 Dimensional Extensions and Other Generalizations

The general result (15) and the procedure modeled on (16)–(17) then explains where are all the twisted variants of supermultiplets in spacetimes of dimension four or more.

**Proof [of Theorem 1.1]:** By applying definition 2.3, all component fields of a given supermultiplet flip their spin-statistics. However, in spacetimes of dimension  $d \geq 3$ , the Lorentz group is non-abelian and the nontrivial irreducible representations have various dimensions, all bigger than one. For  $d \geq 4$ , in all supermultiplets that contain only regular component fields, bosons span tensorial representations of the Lorentz group,  $Spin(1, d-1)$ , while fermions span spinorial representations.

In the twisted (Klein-flipped) variant, the components spanning spinorial (half-integral) Lorentz-representations would have to be bosonic (commuting) fields, and the components spanning tensorial (integral spin) Lorentz-representations would have to be fermionic (anticommuting) fields. The twisting operation modeled on (16)–(17) and specified in definition 2.3 swaps all symmetry group representation assignments of the bosons with those of the fermions, and was employed in Ref. [26]. There, however, the considered group was that of effective/dynamical symmetries rather than the group of Lorentz symmetries, and so did not relate to the spin-statistics correlation. For the same reason, while the well-known triality of  $Spin(8)$  could circumnavigate the above conclusion,  $Spin(8)$  cannot be the Lorentz group in any spacetime.

Therefore, in the twisted (Klein-flipped) variant of any regular (non-ghost) off-shell supermultiplet in any spacetime of dimension  $d \geq 4$ , the component fields span Lorentz-representations of opposite spin-statistics, *i.e.*, *they all are ghosts*. Curiously, the converse of this statement must then also be true: ghost supermultiplets (as needed in supersymmetric gauge theory and/or the “antifield” quantization method, for example) must have the structure of twisted variants of the corresponding regular supermultiplets.  $\checkmark$

**Beyond Adinkras:** The foregoing applies to all adinkraic supermultiplets—those in the worldline dimensional reduction of which each supercharge transforms every component field into precisely one other component field or its derivative.

While the simplest and most often used off-shell supermultiplets are indeed adinkraic, they may be used to construct indefinitely many and ever larger off-shell supermultiplets that are not adinkraic [14], some of which in fact are in current use [28,23].

In particular, Ref. [6] presents a sequence of indefinitely many supermultiplets defined by means of superderivative constraining, generalizing Eqs. (5)–(6). In all such constructions, one starts with an array (direct sum) of real, off-shell intact Salam-Strathdee superfields [11],  $\mathbb{U}_a$ , and defines a superfield/supermultiplet as the solution of the superderivative system

$$\mathbb{A} := \{ \mathbb{U}_a : \sum_a \mathbb{D}_A^a \mathbb{U}_a = 0, \quad \forall A \}, \quad (18)$$

where each  $\mathbb{D}_A^a$  is a suitable matrix of formal multinomials in the superderivatives  $D_I$ ’s. Formally, the space of solutions of (18) is the kernel of the mapping  $\mathbb{D}_A^a : \mathbb{U}_a \rightarrow \tilde{\mathbb{U}}_A$ , the dual of which is the “gauge” equivalence class

$$\mathbb{V} = \tilde{\mathbb{U}}_A / (\mathbb{D}_A^a \mathbb{U}_a) := \{ \phi \in \tilde{\mathbb{U}}_A : \phi \simeq \phi + \mathbb{D}_A^a \varphi, \quad \varphi \in \mathbb{U}_a \}. \quad (19)$$

Since  $\{Q_I, D_J\} = 0$ , such maps are supersymmetric, so that both  $\mathbb{A}$  and  $\mathbb{V}$  are invariantly defined off-shell supermultiplets. Ref. [14] verifies that the so-constructed  $\mathbb{V}$ ’s are most often not



adinkraic. From the standard theory of linear mappings of algebraic structures, we then have the sequence:

$$\{ \ker(\mathbb{D}_A^a) = \mathbb{A} \} \xrightarrow{\iota} \{ \mathbb{U}_a \} \xrightarrow{\mathbb{D}_A^a} \{ \mathbb{V} = \ker(\mathbb{D}_A^a) \} \quad (20)$$

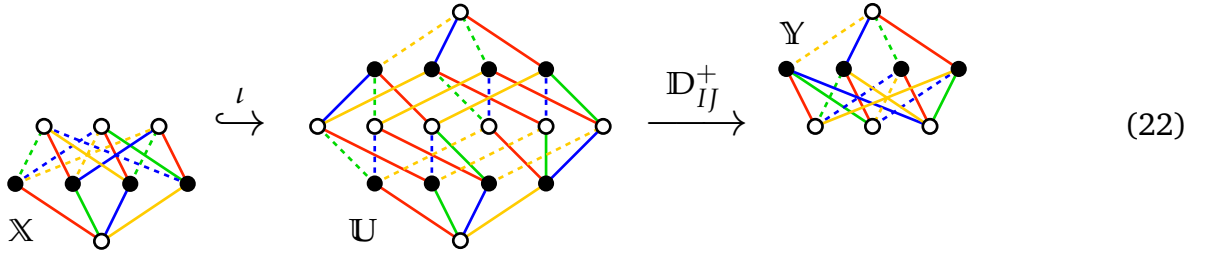
where  $\iota$  is a 1–1 inclusion map and  $\mathbb{D}_A^a \circ \iota = 0$ . One also says that  $\{ \mathbb{U}_a \}$  is then an *extension* of  $\mathbb{V}$  by  $\mathbb{A}$ , which is a non-symmetric notion of a sum.

Since the  $\mathbb{D}_A^a$ -map and the constraint equations (18) are all linear in  $\mathbb{U}_a$ , both  $CP_{\mathbf{c}}(\mathbb{A})$  and  $CP_{\mathbf{c}}(\mathbb{V})$  are complementary, determined by the choice of the  $\mathbb{D}_A^a$ 's, and may be computed along the lines of (11)–(14). It follows that

$$CP_{\mathbf{c}}(\mathbb{A}) + CP_{\mathbf{c}}(\mathbb{V}) = CP_{\mathbf{c}}(\mathbb{U}_a), \quad (21)$$

which vanishes, as each  $\mathbb{U}_a$  is an intact supermultiplet. Presumably, this result may be generalized further, with the  $\mathbb{U}_a$  replaced by non-intact, and even non-Adinkraic supermultiplets.

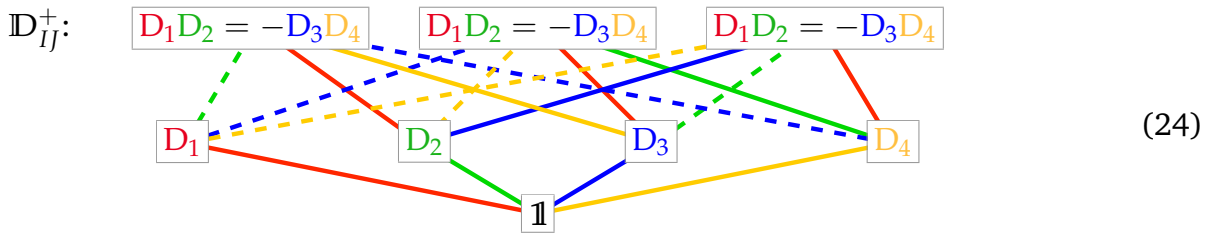
**A Known Example:** The result (21) certainly agrees with the simplest construction of this kind, where superderivative constraints generalizing (12a) and (13a) were used to reduce the intact supermultiplet  $\mathbb{U}$  to a sub-supermultiplet  $\mathbb{A}_{\mathcal{C}}$  of any one of the  $\sim 10^{12}$   $\mathcal{C}$ -encoded chromotopologies [21]. In the simplest nontrivial case, for  $N = 4$ , we have



$$\mathbb{X} \subset \mathbb{U} : \mathbb{D}_{IJ}^+ \mathbb{X} = 0, \quad \text{and} \quad \mathbb{Y} := \{ \mathbb{U} \simeq \mathbb{U} + \iota(\mathbb{X}) \}, \quad (23)$$

using the definition (12b). Following a cycle of, say,  $\mathbf{c} = \{\text{red, green, blue, yellow}\}$  edges starting from any bosonic (white) node, it is easy to compute  $CP_{\mathbf{c}}(\mathbb{X}) = 1$ ,  $CP_{\mathbf{c}}(\mathbb{Y}) = -1$  and  $CP_{\mathbf{c}}(\mathbb{U}) = 0$ .

As stated above, the structure of the  $Q$ -action in  $\mathbb{X}$  is determined by  $\mathbb{D}_{IJ}^+$ , since



The nodes of this *operatorial* Adinkra specifies 1+4+3 operators, and its edges encode the sign in the successive application of these operators. For example, following the left-most edges from  $\mathbb{1}$  upward and acting with the  $D_I$  from the left, we have that

$$\mathbb{1} \xrightarrow[\text{solid}]{+D_1} +D_1 \xrightarrow[\text{dashed}]{-D_2} -D_2 D_1 = +D_1 D_2, \quad (25a)$$

while following the next to left-most edges one has

$$\mathbb{1} \xrightarrow[\text{solid}]{+D_2} +D_2 \xrightarrow[\text{solid}]{+D_1} +D_1 D_2, \quad (25b)$$

next to that,

$$\mathbb{1} \xrightarrow[\text{solid}]{+D_3} +D_3 \xrightarrow[\text{solid}]{+D_4} +D_4 D_3 = -D_3 D_4 \stackrel{(12a)}{=} +D_1 D_2, \quad (25c)$$

and so on. It is easy to see that  $CP_{\{r,g,b,y\}}(\mathcal{C} \in (24)) = +1$ , just as it is for  $\mathbb{X}$ . Since  $\mathbb{U}$  is an intact supermultiplet,  $CP_{\{r,g,b,y\}}(\mathbb{U}) = 0$ , Eq. (21) implies that  $CP_{\{r,g,b,y\}}(\mathbb{Y}) = -1$ , as is easy to verify. This is the precise sense in which  $CP_{\mathbf{c}}(\mathbb{X})$  and  $CP_{\mathbf{c}}(\mathbb{Y})$  are both determined by  $\mathbb{D}_{IJ}^+$  and  $CP_{\mathbf{c}}(\mathbb{U})$ .

In generalizations of the sequence (22) such as studied in Ref. [14],  $\mathbb{X}$  and  $\mathbb{Y}$  are adinkraic only in exceptional cases. However, as long as the central supermultiplet  $\mathbb{U}$  is a direct sum of intact supermultiplets, it will follow that  $CP_{\mathbf{c}}(\mathbb{Y}) = -CP_{\mathbf{c}}(\mathbb{X})$ . This prompts:

**Conjecture 3.1** *Given a short exact sequence<sup>8</sup>  $\mathbb{X} \xrightarrow{\alpha} \mathbb{U} \xrightarrow{\omega} \mathbb{Y}$  of linear supersymmetric maps between off-shell supermultiplets, i.e., given that  $\mathbb{X} : \omega(\mathbb{X}) = 0$ ,  $\omega \circ \alpha = 0$ ,  $\alpha(\mathbb{X}) \approx \mathbb{X}$  and  $\mathbb{Y} := \{\mathbb{U} \simeq \mathbb{U} + \alpha(\mathbb{X})\}$ ,*

$$CP_{\mathbf{c}}(\mathbb{X}) + CP_{\mathbf{c}}(\mathbb{Y}) = CP_{\mathbf{c}}(\mathbb{U}) \quad (26)$$

*for every fixed cycle of distinct edge-colors  $\mathbf{c}$ , for all off-shell supermultiplets  $\mathbb{X}, \mathbb{U}, \mathbb{Y}$  of  $N$ -extended worldline supersymmetry, regardless whether they are adinkraic.*

The relationships indicated in (22) are in fact fairly well-known in the supersymmetry literature in 1+3-dimensional spacetime: The central Adinkra  $\mathbb{U}$  may be identified with the self-conjugate (real), so-called “vector” superfield  $V$ . Identifying then the Adinkra  $\mathbb{X}$  with the superfield combination  $i(\Lambda - \Lambda^\dagger)$  leads us to identifying  $\mathbb{Y}$  with the gauge equivalence class  $V \simeq V + i(\Lambda - \Lambda^\dagger)$  in the Wess-Zumino gauge. The component fields of these supermultiplets obey the standard spin-statistics correspondence, not the “wrong” one of ghosts. Yet,  $\mathbb{X}$  from (22) is related via node-raising/lowering to  $\Phi$  from (8), while  $\mathbb{Y}$  from (22) is similarly related to  $\Xi$  from (8), the twisted variant of  $\Phi$ .

**Theorem 1.1 Compliance:** Nevertheless, and although  $\mathbb{X}$  and  $\mathbb{Y}$  indeed are related via node-raising/lowering to the twisted pair  $\Phi$  and  $\Xi$ , respectively, the display (22) clearly shows that  $\mathbb{Y}$  is *not* the twisted variant of  $\mathbb{X}$ : their nodes are positioned at different heights.

Thus, although the worldline supermultiplets

$$\begin{aligned} \Phi &= (\phi_1, \phi_2 | \psi_I | F_3, F_4), \quad (\phi_i | \psi_I), \quad \text{and} \quad \mathbb{X} = (\phi_1 | \psi_I | F_3, \dot{\phi}_2, F_4), \\ \text{where } \phi_3 &:= (\int d\tau F_3), \quad \text{and} \quad \phi_4 := (\int d\tau F_4), \end{aligned} \quad (27)$$

are all related to each other by means of node-raising/lowering and so have the same supersymmetry transformation structure<sup>9</sup>, they are:

1. locally inequivalent worldline supermultiplets, and also
2. worldline reductions of distinct higher-dimensional supermultiplets—if they extend at all to higher-dimensional spacetime [17,18,19,15].

<sup>8</sup> That  $\mathbb{X} \xrightarrow{\alpha} \mathbb{U} \xrightarrow{\omega} \mathbb{Y}$  is a “short exact sequence” means that:  $\alpha$  is a 1–1 injection,  $\omega$  is a surjection and  $\omega \circ \alpha = 0$ , so that  $(\mathbb{X} \simeq \text{im}(\alpha) \subset \mathbb{U}) = \ker(\omega)$  and  $\mathbb{Y} = \text{cok}(\alpha) = \{\mathbb{U} / \text{im}(\alpha)\}$ .

<sup>9</sup> Following the formalism of Refs. [8,9,10], this structure might be called *dashed chromotopology*.

Indeed, this applies just as well to all the sixteen node-raised/lowered variants of (27): starting with  $(\phi_i|\psi_I)$ , one can “raise” any of the four bosons, obtaining  $2^4 = 16$  Adinkras with different node-height arrangements. The analogous applies also to the twisted variants of (27):

$$\Xi = (x_1, x_2|\xi_I|X_3, X_4), \quad (x_i|\xi_I), \quad \text{and} \quad \Upsilon = ((-x_1), x_2, x_3|-\xi_1, \xi_2, \xi_3, \xi_4|X_4), \quad (28)$$

where  $x_3 := (\int d\tau X_3)$ , and  $x_4 := (\int d\tau X_4)$

and the remaining thirteen node-raised/lowered versions of these.

Finally, since  $\mathbb{X} = (\phi_1|\psi_I|F_2, F_3, F_4)$  is the worldline reduction of the supermultiplet  $i(\Lambda - \Lambda^\dagger)$  of regular component fields, theorem 1.1 guarantees that its twisted variant,

$$\tilde{\mathbb{X}} = (\phi_1^K|\psi_I^K|F_2^K, F_3^K, F_4^K) \xleftrightarrow{(16)} (\xi_2|X_3, \dot{x}_1, X_4, \dot{x}_2|\dot{\xi}_4, \dot{\xi}_1, \dot{\xi}_3) \quad (29)$$

is indeed the worldline reduction of a supermultiplet in a higher dimensional spacetime (such as the 1+3-dimensional one), the component fields of which are all ghosts:  $\phi_1^K$  and  $F_3^K$  are anticommuting scalars,  $\dot{\phi}_2^K$  and  $F_4^K$  anticommuting pseudo-scalars, and  $\psi_I^K$  are commuting spinors. The  $Q$ -action amongst these ghost component fields is identical to the one within the supermultiplet  $(\xi_2|X_3, \dot{x}_1, X_4, \dot{x}_2|\dot{\xi}_1, \dot{\xi}_4, \dot{\xi}_3)$ , which has in turn been manifestly obtained from the twisted-chiral supermultiplet depicted on the right-hand side of (8) by means of node-raising. Note that:

1.  $(\xi_2|X_3, \dot{x}_1, X_4, \dot{x}_2|\dot{\xi}_1, \dot{\xi}_4, \dot{\xi}_3)$  can extend to a worldsheet supermultiplet, with the Lorentz group  $Spin(1, 1)$ . However, owing to the obstruction described in Ref. [19], this can only be a supermultiplet of worldsheet (1, 3)- or (3, 1)-supersymmetry [15].
2.  $(\xi_2|X_3, \dot{x}_1, X_4, \dot{x}_2|\dot{\xi}_1, \dot{\xi}_4, \dot{\xi}_3)$  cannot extend to a  $(d > 2)$ -dimensional spacetime: already for  $d = 3$ , the irreducible spinors of  $Spin(1, 2)$  have two components, and it is impossible to Lorentz-covariantly separate the spinor component  $\xi_2$  from the  $\dot{\xi}_1, \dot{\xi}_4, \dot{\xi}_3$ .
3. In turn, the ghost supermultiplet  $(\phi_1^K|\psi_I^K|F_3^K, \dot{\phi}_2^K, F_4^K)$  straightforwardly extends to 1+2- and also to 1+3-dimensional spacetime.

Similarly,  $\Upsilon = ((-x_1), x_3, x_2|-\xi_1, \xi_2, \xi_3, \xi_4|X_4)$  is the worldline reduction of the supermultiplet  $V \simeq V + i(\Lambda - \Lambda^\dagger)$  of regular component fields in the Wess-Zumino gauge, where the component fields of  $i(\Lambda - \Lambda^\dagger)$  are used to cancel component fields in  $V$ . Then,

$$\tilde{\Upsilon} = ((-x_1^K), x_3^K, x_2^K|-\xi_1^K, \xi_2^K, \xi_3^K, \xi_4^K|X_4^K) \xrightarrow{*} (-\psi_2, \psi_1, \psi_4|-F_3, \dot{\phi}_1, F_4, \dot{\phi}_2|\dot{\psi}_3) \quad (30)$$

where “\*” denotes the correspondence (16) followed however by a subsequent application of  $\partial_\tau$  on every component field in the right-hand side supermultiplet. Again,  $\tilde{\Upsilon}$  is the worldline reduction of a supermultiplet in a higher dimensional spacetime, the component fields of which are all ghosts. Amongst these, the  $Q$ -action is identical to the one in the supermultiplet  $(-\psi_2, \psi_1, \psi_4|-F_3, \dot{\phi}_1, F_4, \dot{\phi}_2|\dot{\psi}_3)$ , which has in turn been manifestly obtained from the chiral supermultiplet depicted on the left-hand side of (8) by means of node-raising. In turn, the regular (non-ghost) supermultiplet  $(-\psi_2, \psi_1, \psi_4|-F_3, \dot{\phi}_1, F_4, \dot{\phi}_2|\dot{\psi}_3)$  cannot extend to spacetimes beyond  $d = 2$  since  $\dot{\psi}_3$  can be separated from  $\psi_1, \psi_2, \psi_4$  in a Lorentz-covariant way only for  $d \leq 2$ . In fact, even on the worldsheet, this can only be a supermultiplet for (3, 1)- or (1, 3)-supersymmetry [15, 19].

## 4 Conclusions

It has been shown herein that in all supersymmetric models with spacetime of four or more dimensions, the type of supermultiplet twisting discovered in Ref. [1] and first employed in Ref. [2], if nontrivial, necessarily maps regular (non-ghost) supermultiplets to ghost supermultiplets. The precise conditions for when this type of twisting is non-trivial may be found in Refs. [8,9], and include the restriction that the total number of supercharge components must be divisible by four—which is true in all ( $d \geq 4$ )-dimensional spacetime.

The particular implementation of this supermultiplet twisting has been shown to include a (boson  $\leftrightarrow$  fermion) Klein-flip; see section 2 and also Ref. [26]. Regular (non-ghost) fields obey the standard spin-statistics correspondence, so bosons span tensorial representations of the Lorentz group while fermions span spinorial representations; ghost fields obey the flipped, “wrong” spin-statistics correspondence. The supermultiplet twisting (Klein-flip) therefore maps regular (non-ghost) component fields to ghost component fields and *vice versa*.

This conclusion can be avoided only in spacetimes of low enough dimension, *i.e.*, on the worldsheet and the worldline, where the Lorentz group  $Spin(1, d-1)$  is abelian, the irreducible tensorial and spinorial representations are all 1-dimensional and therefore interchangeable.

While it is true that there exist exceptional Lie groups—such as  $Spin(8)$ —for which the tensorial and spinorial representations are interchangeable without being 1-dimensional, there exists no such  $Spin(1, d-1)$  Lorentz group. As observed in Ref. [26], the representations spanned by the bosons and the fermions are additionally restricted by the requirements

$$\mathcal{R}_L(Q) \otimes \mathcal{R}_L(\phi) \supset \mathcal{R}_L(\psi) \quad \text{and} \quad \mathcal{R}_L(Q) \otimes \mathcal{R}_L(\psi) \supset \mathcal{R}_L(\phi), \quad (31)$$

where  $\mathcal{R}_L(Q)$  for our present purposes denotes the representation of the Lorentz group spanned by the supercharges, while  $\mathcal{R}_L(\phi)$  and  $\mathcal{R}_L(\psi)$  denote the Lorentz-representation spanned by the bosonic and fermionic component fields, respectively. For the regular spin-statistics correspondence,  $\mathcal{R}_L(\phi)$  must be a tensorial representation while  $\mathcal{R}_L(\psi)$  must be a spinorial one; for the “wrong” spin-statistics correspondence in ghost supermultiplets, this is reversed. However, the Haag-Łopuszański-Sohnius theorem guarantees that  $\mathcal{R}_L(Q)$  must in all circumstances be a spin- $1/2$  Lorentz-representation.

**Acknowledgments:** Many thanks to Mr. James Parker for the advance copy of Ref. [20], written under the mentorship of Prof. G.D. Landweber, as well as to S.J. Gates, Jr., for many helpful discussions. I am grateful to the Department of Energy for the generous support through the grant DE-FG02-94ER-40854, as well as the Physics Department of the Faculty of Natural Sciences of the University of Novi Sad, Serbia, for recurring hospitality and resources.

## References

- [1] S. J. Gates, Jr., *Superspace formulation of new nonlinear sigma models*, *Nucl. Phys.* B238 (1984) 349–366.
- [2] S. J. Gates, Jr., C. M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear sigma models*, *Nucl. Phys.* B248 (1984) 157.
- [3] T. Hübsch, *Linear and chiral superfields are usefully inequivalent*, *Class. Quant. Grav.* 16 (1999) L51–L54. <http://arXiv.org/abs/hep-th/9903175>
- [4] T. Hübsch, *Haploid (2,2)-superfields in 2-dimensional space-time*, *Nucl. Phys.* B555(3) (1999) 567–628.
- [5] R. Q. Almukahhal and T. Hübsch, *Gauging Yang-Mills symmetries in 1+1-dimensional space-time*, *Int. J. Mod. Phys. A* 16(29) (2001) 4713–4768.
- [6] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga and G. D. Landweber, *On graph-theoretic identifications of Adinkras, supersymmetry representations and superfields*, *Int. J. Mod. Phys. A* 22 (2007) 869–930. <http://arxiv.org/abs/math-ph/0512016>
- [7] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga and G. D. Landweber, *Relating doubly-even error-correcting codes, graphs, and irreducible representations of  $N$ -extended supersymmetry*, in: *F. Liu and et al. (Eds.), Discrete and Computational Mathematics*, Nova Science Publishers, Inc., Hauppauge, NY, 2008. <http://arxiv.org/abs/0806.0051>
- [8] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga, G. D. Landweber and R. L. Miller, *Topology types of Adinkras and the corresponding representations of  $N$ -extended supersymmetry*. <http://arxiv.org/abs/0806.0050>
- [9] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga, G. D. Landweber and R. L. Miller, *Adinkras for clifford algebras, and worldline supermultiplets*. <http://arxiv.org/abs/0811.3410>
- [10] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga, G. D. Landweber and R. L. Miller, *Codes and supersymmetry in one dimension*, *Adv. Theor. Math. Phys.* (accepted for publication). <http://arxiv.org/abs/1108.4124>
- [11] A. Salam and J. Strathdee, *On superfields and Fermi-Bose symmetry*, *Phys. Rev. D* 11 (1975) 1521–1535.
- [12] S. J. Gates, Jr., M. T. Grisaru, M. Roček and W. Siegel, *Superspace*, Benjamin/Cummings Pub. Co., Reading, MA, 1983.
- [13] I. L. Buchbinder and S. M. Kuzenko, *Ideas and methods of supersymmetry and supergravity*, *Studies in High Energy Physics Cosmology and Gravitation*, IOP Publishing Ltd., Bristol, 1998.
- [14] T. Hübsch and G. Katona, *On the construction and the structure of off-shell supermultiplet quotients*. <http://arxiv.org/abs/1202.4342>
- [15] T. Hübsch, *Weaving worldsheet supermultiplets from the worldlines within*. <http://arxiv.org/abs/1104.3135>

- [16] T. Hübsch, *Superspace: A vast algebraic variety*, in: L. Ji (Ed.), *Geometry and Analysis*, Vol. 2 of *Advanced Lectures in Mathematics*, International Press, 2010, pp. 39–67, closing address at the Conference “Geometric Analysis: Present and Future”, Harvard University, 2008.  
<http://arxiv.org/abs/0901.2136>
- [17] M. G. Faux, K. M. Iga and G. D. Landweber, *Dimensional enhancement via supersymmetry*.  
<http://arxiv.org/abs/0907.3605>
- [18] M. G. Faux and G. D. Landweber, *Spin holography via dimensional enhancement*, *Phys. Lett. B* 681 (2009) 161–165. <http://arxiv.org/abs/0907.4543>
- [19] S. J. Gates, Jr. and T. Hübsch, *On dimensional extension of supersymmetry: From worldlines to worldsheets*. <http://arxiv.org/pdf/1104.0722>
- [20] J. Parker, *Diamonds and bowties: Supersymmetric graph conditions*, completed at the Mathematic REU program at Bard College during the Summer of 2011.
- [21] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga and G. D. Landweber, *A superfield for every dash-chromotopology*, *Int. J. Mod. Phys. A* 24 (2009) 5681–5695.  
<http://arxiv.org/abs/0901.4970>
- [22] S. J. Gates, Jr., J. Gonzales, B. MacGregor, J. Parker, R. Polo-Sherk, V. G. J. Rodgers and L. Wassink, *4D,  $\mathcal{N} = 1$  supersymmetry genomics (I)*, *JHEP* 12 (2009) 009. <http://arXiv.org/abs/0902.3830>
- [23] S. J. Gates, Jr., J. Hallet, T. Hübsch and K. Stiffler, *The real anatomy of complex linear superfields*.  
<http://arxiv.org/abs/1202.4418>
- [24] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga and G. D. Landweber, *An application of cubical cohomology to Adinkras and supersymmetry representations*. [forthcoming](#)
- [25] S. J. Gates, Jr. and L. Rana, *A theory of spinning particles for large  $N$ -extended supersymmetry. II*, *Phys. Lett. B* 369(3-4) (1996) 262–268. <http://arxiv.org/abs/hep-th/9510151>
- [26] M. G. Faux, S. J. Gates, Jr. and T. Hübsch, *Effective symmetries of the minimal supermultiplet of  $N = 8$  extended worldline supersymmetry*, *J. Phys. A* 42 (2009) 415206. <http://arxiv.org/abs/0904.4719>
- [27] S. J. Gates, Jr. and L. Rana, *Ultramultiplets: A new representation of rigid 2-d,  $N=8$  supersymmetry*, *Phys. Lett. B* 342 (1995) 132–137. <http://arxiv.org/abs/hep-th/9410150>
- [28] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga and G. D. Landweber, *Frames for supersymmetry*, *Int. J. Mod. Phys. A* 24 (2009) 2665–2676. <http://arxiv.org/abs/0809.5279>